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A Characterization of Gaussian Spaces

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A characterization of Gaussian subspaces of L_2 is given. It is shown that Gaussian subspaces are essentially characterized by the property that all linear isometries are induced by measure-preserving transformations.

Let T be a measure-preserving transformation acting on a probability space $(\Omega, \mathscr{F}, P)$. This T induces a linear isometry V on $L_2(\Omega, \mathscr{F}, P)$ defined by $V(f) = f \circ T$. The analysis of T by using properties of V is referred to as spectral analysis. It is often profitable to consider the action of V on certain closed linear invariant subspaces $\mathscr{L}$ of $L_2(\Omega, \mathscr{F}, P)$. In this note we make the following assumptions about $\mathscr{L}$:

1. $\mathscr{L}$ generates $\mathscr{F}$, i.e., $\mathscr{F}$ is the smallest σ-field $\mathscr{G}$ such that f is $\mathscr{G}$-measurable whenever $f \in \mathscr{L}$.
2. $\int_\Omega f\,dP = 0$ for all $f \in \mathscr{L}$.
3. $\mathscr{L}$ is infinite dimensional.

Condition (1) is necessary if V restricted to $\mathscr{L}$ is to contain any information about T, and (2) removes the redundant information that the constant functions are invariant under all operators induced by transformations. Condition (3) is discussed in Remark 1, at the end.

Very detailed information is available when $\mathscr{L}$ is a Gaussian space, cf. Kakutani (1960) and Veršik (1962). $\mathscr{L}$ is said to be a Gaussian space if every non-zero function in $\mathscr{L}$ has a normal distribution. The following two facts make Gaussian spaces particularly well suited for studying measure-preserving transformations.

A. If two measure-preserving transformations induce unitarily equivalent operators on $\mathscr{L}$, then they are isomorphic. (This is so because the normal distribution is determined by its first two moments.)

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B. Every linear isometry on $\mathscr{L}$ is induced by a measure-preserving transformation (cf. Wiener and Akutowicz (1957)).

The purpose of this note is to consider the converse of (B). We shall also give a slightly simpler proof of (B) than that given by Wiener and Akutowicz (1957).

THEOREM. *Let $\mathscr{L}$ satisfy assumptions* 1, 2 *and* 3 *above. Then $\mathscr{L}$ is a Gaussian space if and only if every linear isometry on $\mathscr{L}$ is induced by a measure-preserving transformation and at least one of these transformations is ergodic.*

Suppose $\mathscr{L}$ is a Gaussian space and let $\{X_\lambda: \lambda \in \Lambda\}$ be an orthonormal basis of $\mathscr{L}$. We first replace $(\Omega, \mathscr{F}, P)$ by its Kolmogorov representation. Let $\Omega = \mathbb{R}^\Lambda$, where $\mathbb{R}$ denotes the real line. Let $\tilde{\mathscr{F}}$ be the σ-field of usual Borel sets on $\tilde{\Omega}$. Then $\varphi: \Omega \to \tilde{\Omega}$ defined by $\varphi(\omega)_\lambda = X_\lambda(\omega)$ is an $\mathscr{F} - \tilde{\mathscr{F}}$ measurable map and we define $\tilde{P}$ on $\tilde{\mathscr{F}}$ by $\tilde{P}(\varphi^{-1}(A))$. Finally define $\tilde{X}_\lambda$ by $\tilde{X}_\lambda(\tilde{\omega}) = \tilde{\omega}_\lambda$ and the $\tilde{\mathscr{L}}$ be the subspace of $L_2(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ generated by $\{\tilde{X}_\lambda: \lambda \in \Lambda\}$. It follows that $\mathscr{L}$ and $\tilde{\mathscr{L}}$ have exactly the same probability structure. We will prove the theorem on $\tilde{\Omega}$ since that is what we really mean by induced transformation. Let V be a linear isometry on $\mathscr{L}$, and define $R: \tilde{\Omega} \to \tilde{\Omega}$ by the equation $T(\omega)_\lambda = V(\tilde{X}_\lambda)(\omega)$. V is clearly induced by T and it remains to show that T is measure preserving. But for $\lambda_1, \ldots, \lambda_n \in \Lambda$ and $B_1, \ldots, B_n$ real Borel sets

$$T^{-1} \bigcap_{j=1}^{n} [\tilde{X}_{\lambda_j} \in B_j] = \bigcap_{j=1}^{n} [V(\tilde{X}_{\lambda_j}) \in B_j].$$

Since V is a linear isometry, the random variables $\{\tilde{X}_{\lambda_j}\}$ and $\{V(\tilde{X}_{\lambda_j})\}$ have the same covariance structure. Since the random variables are Gaussian, this implies that the probabilities are the same:

$$\tilde{P}\left(T^{-1} \bigcap_{j=1}^{n} [\tilde{X}_{\lambda_j} \in B_j]\right) = \tilde{P}\left(\bigcap_{j=1}^{n} [V(\tilde{X}_{\lambda_j}) \in B_j]\right)$$
$$= \tilde{P}\left(\bigcap_{j=1}^{n} [\tilde{X}_{\lambda_j} \in B_j]\right).$$

Thus T is measure-preserving. Since $\mathscr{L}$ is infinite dimensional, there exists an orthogonal basis $\{Z_1, Z_2, \ldots\}$. By the Gaussian property, these are independent and identically distributed and the transformation that is induced by the linear isometry that sends Z_n to Z_{n+1} is clearly ergodic.

Suppose conversely that every linear isometry on $\mathscr{L}$ is induced by a measure-preserving transformation. Let $\{X_1, X_2, \ldots\}$ be an orthonormal basis in $\mathscr{L}$. Since X_1 is an arbitrary normalized function in $\mathscr{L}$, it is sufficient to show that X_1 has a normal distribution. Let $\varphi(t) = \int_\Omega \exp(itX_1)\, dP$ be the characteristic function of X_1. Since every unitary operator on the space generated by $\{X_1,..., X_n\}$ is induced by a measure-preserving transformation, the joint characteristic function of $\{X_1,..., X_n\}$, $\varphi_n(\mathbf{t}) = \int_\Omega \exp(i \sum t_j X_j)\, dP$, is given by $\varphi_n(\mathbf{t}) = \varphi(\|\mathbf{t}\|)$, where $\|\mathbf{t}\|$ is the Euclidean norm of $\mathbf{t} = (t_1,..., t_n)$. To see this consider a unitary operator that sends $\sum t_j X_j$ to $\|\mathbf{t}\| X_1$. Since for all n, $\varphi(\|\mathbf{t}\|)$ is a non-negative definite function on $\mathbb{R}^n$, Theorem 2 of Schoenberg (1938) states that $\varphi(t) = \int_0^\infty \exp(-t^2\sigma^2/2)\, d\mu(\sigma)$, where μ is a Borel probability measure on $\mathbb{R}^+$. Now let $Y, Y_1, Y_2,...$ be a sequence of independent random variables where, Y has distribution μ and each Y_k is normal with mean 0 and variance 1. Set $Z_n = Y \cdot Y_n$. A simple calculation shows that $\{Z_1,..., Z_n\}$ has the same characteristic function as $\{X_1,..., X_n\}$. Since the characteristic functions uniquely determine the distributions we may suppose that $X_n = Y \cdot Y_n$ for all n. Taking linear combinations and L_2 limits, we see that $\mathscr{L}$ has the form of $Y \cdot \hat{\mathscr{L}}$, where $\hat{\mathscr{L}}$ is Gaussian and Y is an independent random variable with distribution μ. Let V be a linear isometry on $\mathscr{L}$ induced by an ergodic measure-preserving transformation T. We shall complete the proof by showing that Y is equal to 1 almost everywhere. Since $\mathscr{L}$ is invariant under V, the set $[Y=0]$ is invariant under T and hence has probability zero. Thus for U in $\hat{\mathscr{L}}$, W defined on $\hat{\mathscr{L}}$ by $W(U) = V(YU)/Y$ is well defined and may be shown to be a linear isometry since $\{Y \cdot Y_n\}$ and $\{Y_n\}$ are both orthonormal bases. Since $\hat{\mathscr{L}}$ is Gaussian W is induced by a measure-preserving transformation S. Since Y is stochastically independent of $\hat{\mathscr{L}}$ we may suppose $Y \circ S = Y$. Now we claim that W has no non-zero eigenvectors in $\hat{\mathscr{L}}$. Suppose, on the contrary, f is such an eigenvector for W. Then $Y \cdot f$ is an eigenvector for V, which implies that $Y \cdot |f|$ is invariant under T. But since T is ergodic and invariant, this means $Y \cdot |f|$ is a constant a.s. This contradicts the fact that Y and $|f|$ are independent with $|f|$ being unbounded (since f is normal). Since W has no eigenvectors on $\hat{\mathscr{L}}$, S is ergodic on the σ-field generated by $\hat{\mathscr{L}}$ (see, for instance, Wiener and Akutowicz (1957), Theorem 1.) An induction argument yields $V^n(YU) = Y \cdot W^n(U)$. Assuming without loss of generality that $\int_\Omega U^2\, dP = 1$,

$$
\begin{aligned}
1 = \int_\Omega Y^2 U^2\, dP &= \lim(1/n) \sum_{j=1}^{n} (YU)^2 \circ T^j \\
&= Y^2 \lim(1/n) \sum_{j=1}^{n} U_2 \circ S^j \\
&= Y_2.
\end{aligned}
$$

Thus $Y = 1$ almost surely, as desired. ∎

Remark 1. If $\mathscr{L}$ is finite dimensional, the distributions that have the property that every linear isometry is induced by a measure-preserving transformation, are the spherically symmetry distributions. These have been studied extensively and it is known that this class, while being wider than the spherically symmetric Gaussian distributions, has many properties in common with the latter. See, for instance, Lord (1954) and Kelker (1970).

Remark 2. Let $\hat{\mathscr{L}}$ be any Gaussian space and set $\mathscr{L} = Y \cdot \hat{\mathscr{L}}$, where Y is a non-negative square integrable random variable, independent of all the random variables in $\hat{\mathscr{L}}$. Then the above proof shows that $\mathscr{L}$ also has the property that every linear isometry on $\mathscr{L}$ is induced by a measure-preserving transformation. $\mathscr{L}$ will be Gaussian if and only if Y is a constant almost surely. Since $\mathscr{L}$ and $\hat{\mathscr{L}}$ are indistinguishable as Hilbert spaces, it follows that some extra condition would be needed. Ergodicity is perhaps the most natural such condition for a measure-preserving transformation and our theorem is a confirmation of this.

Remark 3. It is well known that ergodicity and the property of weakly mixing are equivalent for Gaussian spaces (cf., for example, Wiener and Akutowicz (1957), Theorems 1 and 6). But, in general, weakly mixing is a stronger condition than ergodicity.

References

[1] Lord, R. (1954). The use of Hankel transforms in statistics, I, general theory and examples. *Biometrika* **41** 44–55.

[2] Kakutani, S. (1960). Spectral analysis of stationary Gaussian processes. In *Proceedings, 4th Berkeley Symp. Stat. and Prob.*, Vol. 2, pp. 239–247.

[3] Kelker, D. (1970). Distribution theory of spherical distributions and a location-scale parameter generalization. *Sankhya, Ser. A* 419–430.

[4] Schoenberg, I. (1938). Metric spaces and completely monotone functions. *Ann. of Math.* **39** 811–841.

[5] Veršik, A. (1962). The theory of normal dynamical systems. *Soviet Math. Dokl.* **3** 625–628.

[6] Veršik, A. (1962). Spectral and metric isomorphisms of some normal dynamical systems. *Soviet Math. Dokl.* **3** 693–696.

[7] Wiener, N. and Akutowicz, E. (1957). The definition and ergodic properties of the stochastic adjoint of a unitary transformation. *Rend. Circ. Mat. Palermo* (2) **6** 205–217.